Individual

Problem 1. Let $\phi : (x, y) \in U \subset \mathbb{C} \to (u, v) \in \mathbb{C}$ be a smooth orientation preserving mapping near $P \in U$, i.e. $\det(D\phi(P)) > 0$, where

$D\phi(P) = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$.

Define $\phi_{\bar{z}} = \frac{1}{2}(\phi_x + i\phi_y), \phi_z = \frac{1}{2i}(\phi_x - i\phi_y)$. Show that:

a) $\mu := \frac{|\phi_{\bar{z}}|}{|\phi_z|}(P) < 1$.

b) $D\phi(P)$ maps a circle to an ellipse. μ is the ratio of length of the short axe with the length of the long axe.

Problem 2. Complete the following two parts:

(i) Consider the following Cauchy problem:

$$\begin{cases} u_t - (a(x)u_x)_x = 0 & \text{for } x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = u_0(x), \end{cases} \tag{1}$$

where $a(x) \in C^2$ is bounded with $a(x) \geq a_0 > 0$.

(a) Let $u(x, t)$ be the solution of the Cauchy problem with the following initial data:

$$u_0(x) = \begin{cases} 1 & \text{for } x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Does $u(x, t)$ have zero points on the line $t = \frac{1}{1000}$? Explain your reasons.

(b) Is it possible to prepare some bounded initial data $u_0(x)$ such that the solution $u(t, x)$ becomes

$$u(x, 1) = \begin{cases} 1 & \text{for } x \in [-1, 1], \\ 0 & \text{otherwise,} \end{cases}$$

at $t = 1$? Explain your reasons.

(ii) Consider the following Cauchy problem:

$$\begin{cases} u_{tt} - (a(x)u_x)_x = 0 & \text{for } x \in \mathbb{R},\ t > 0, \\ u|_{t=0} = u_0(x), \quad u_t|_{t=0} = 0. \end{cases} \tag{2}$$

where $a(x) \in C^2$ is bounded with $a(x) \geq a_0 > 0$. Let $u(x,t)$ be the solution of the Cauchy problem with the following initial data:

$$u_0(x) = \begin{cases} 1 & \text{for } x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Does $u(x,t)$ have zero points on the line $t = 1000$? Explain your reasons.

Problem 3. Let $B \subset \mathbb{R}^n$ denote the open unit ball, $n \geq 2$. Assume that $v \in C^\infty(\overline{B})$ and $u \in C^\infty(\overline{B} \setminus \{0\})$ satisfy

$$\begin{aligned} \Delta u &= 0 \quad \text{in } B \setminus \{0\}, \\ u &> 0 \quad \text{in } B \setminus \{0\}, \\ \Delta v &= 0 \quad \text{in } B, \\ u &= v \quad \text{on } \partial B. \end{aligned}$$

Prove that $u \geq v$ on $B \setminus \{0\}$.

Problem 4. Let $\phi \in C_c^\infty(\mathbb{R})$ be a compactly supported smooth function.

a) Prove that

$$\lim_{\lambda \to +\infty} \int_{-\infty}^{+\infty} e^{i\lambda x^2} \phi(x)\, \mathrm{d}x = 0.$$

b) Prove that there are constants $\alpha > 0$ and $C > 0$ such that

$$\forall \lambda \geq 1, \quad \left| \int_{-\infty}^{+\infty} e^{i\lambda x^2} \phi(x)\, \mathrm{d}x \right| \leq C\lambda^{-\alpha}.$$

The constant C may depend on ϕ.
